On the Euler Scale and the μEuclidean Integer Relation Algorithm

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The equal–tempered 10–tone scale $e^{n/10}$ ($n = 0, \pm1, \pm2, \dots$), using the Euler number $e = 2.71828\dots$ as a pseudo–octave is shown to approximate well the prime number harmonics 2, 3, 5, and 11. Equal–tempered scales simultaneously approximating certain frequency ratios, shall be called *tonal scales*.[1] Some of the properties of the Euler scale and its relation to other tonal scales are explored.

The general mathematical problem of identifying tonal scales can be solved by investigating integer relations, using the μEuclidean algorithm, a modification of the PSLQ algorithm. If restricted to two numbers the μEuclidean algorithm goes over identically into the ancient Euclidean algorithm, contrary to the PSLQ algorithm. The μEuclidean algorithm is able to solve a certain class of higher dimensional integer relations where the PSLQ (not the PPSLQ) algorithm breaks down. In general, the μEuclidean algorithm finds smaller integer relations than the (P)PSLQ algorithm.

In an appendix a simple alternative procedure is presented for determining tonal scales based on continued fractions.

Keywords: equal–tempered scales; integer relations; continued fractions; harmonics; intervals

1. Introduction

Contemporary microtonal composers are intrigued by principles that can be used to create novel music. In the construction of musical scales a traditional principle is to seek for a concurrence of structure and consonance. The simplest structural type of musical scales are sequences of pitches or equivalently, frequencies. Other types of musical scales are beyond the scope of this paper. The concept of *consonance* is often used by musicians with the intention to simultaneously characterize perceptual and musical properties of sounds and to thereby bridge the psycho–physical and musical levels in search of new aesthetic phenomena. The psycho–physical concept of

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[1]In this particular meaning, the term is used among microtonal composers.

sensory dissonance is based on dissonance curves which are continuous functions of the frequency ratio between two sounds with fixed timbres. The timbres of the sounds have a significant impact on the shapes of these curves. Hermann von Helmholtz demonstrated that the dissonance curves of harmonic sounds have local minima at simple frequency ratios.

The musically motivated perception and evaluation of consonance changed in the course of time. For example, more and more simple fractions of the fundamental frequency were considered as consonant which, finally, motivated Schönberg to develop his concept of the ‘emancipation of dissonance’ [1].

The usual equal–tempered scale takes the octave as the fundamental unit and divides it into 12 parts according to the formula

$$f = f_0\, 2^{n/12}, \quad n = 0, \pm 1, \pm 2 \ldots,$$

where f_0 is some fundamental frequency, e.g. $f_0 = 440$ Hz. The numbers n are called *intervals*. The *octave* is characterized by the interval $n = 12$, the *unison* by $n = 0$.

Almost hit consonances are related to nearly integer numbers n. E.g. the 3rd harmonic ($f/f_0 = 3$) corresponds approximately to the number $n = 19$ since $12 \log_2 3 = 19.02\ldots$. The fifth ($f/f_0 = 3/2$) is the difference between the 3rd and 2nd harmonic and is well represented by the interval $n = 7$, as follows from $12 \log_2(3/1) - 12 \log_2(2/1) = 12 \log_2(3/2) = 7.02\ldots$.

The Bohlen–Pierce scale [2], [3]

$$f = f_0\, 3^{n/13}, \quad n = 0, \pm 1, \pm 2 \ldots$$

is a non–octave scale. The 3rd harmonic is represented exactly by the integer valued interval $n = 13$. Pierce called the frequency ratio 3/1 a *tritave*. The exponent $n/13$ is fixed by the condition that the 5th harmonic can be approximated by a ‘simple’ ratio in the exponent. Indeed $\log_3 5 = \frac{6}{13} \times 1.007\ldots$ Surprisingly the Bohlen–Pierce scale yields also an excellent approximation of the 7th harmonic as can be read off from $\log_3 7 = \frac{10}{13} \times 1.003\ldots$. However, there is a price to be paid as octaves are imperfectly represented due to $13 \log_3 2 = 8.202\ldots$ or $\log_3 2 = \frac{8}{13} \times 1.025\ldots$.

For the current status about the research on the Bohlen–Pierce scale see [4]. At this conference the first Bohlen–Pierce panflute [5] and the Bohlen–Pierce soprano clarinet [6] were presented.

Within the well–tempered scale the major third and the harmonic seventh are rather poorly approximated by integer values of n because $12 \log_2(5/4) = 3.86\ldots$ and $12 \log_2(7/4) = 9.69\ldots$.

In the composition ‘Partch Harp’ for microtonal acoustic harp and microtonal synthesizer [7], the non–octaving scale

$$f = f_0\, 1.9560685^{n/12}, \quad n = 0, \pm 1, \pm 2 \ldots$$

is used. This choice yields nearly exact integer representations of the major third and the harmonic seventh[2]

$$\frac{5}{4} = 1.9560685^{3.991.../12}, \quad \frac{7}{4} = 1.9560685^{10.009.../12}$$

by the intervals 4 and 10.[3] The octave is 38 cents short [$1200 \log_2 s$ cent $= 1162$ cents] and the fifth 24 cents short [$1200 \log_2 s^{7/12}$ cent $= 678$ cents $= (702 - 24)$ cents]. The deviations are well audible. For octaves, even very small deviations of only 1 or 2 cents may be recognized as distuned. The overall tuning period of the harp are true octaves (2/1), but within every octave, the intervals are tempered according to the 'reduced octave' ($s/1$) and, thus, the last step of the scale is enlarged by a 'Stahnke comma'. In contrast, the synthesizer is completely tuned in 'reduced octaves' ($s/1$). The tunings drift apart, the higher or the lower the register is. The resulting harmonic strangeness contributes significantly to the very special sound of Partch Harp. The well–tuned major thirds and harmonic seventh are noticeable in a unique way and add a most remarkable flavour to the piece.

The Bohlen–Pierce scale can be understood as a fractional 12–tone equal–temperament (ET) scale. Because of the good approximation $2^{19/12} = 2.9966\ldots \simeq 3$, a Stahnke–like representation

$$3^{\frac{n}{13}} \approx (2^{19/12})^{\frac{n}{13}} = 2^{\frac{19}{13}\frac{n}{12}} = 2.7540\ldots^{\frac{n}{12}}$$

is obtained.[4]

Despite these mathematical manipulations, it must not be forgotten that the acoustic properties of a tonal scale do not depend on its mathematical representation but on its harmonic content.

The scales developed by Bohlen–Pierce and Stahnke raised our interest in the problem of tuning non–octave intervals. More specifically, we are looking for algorithms determining intervals subject to several simultaneous 'constraints'. The case of two constraints is well understood. Using the theory of continued fractions it is easily shown that fifths are better approximated within an octave if the octave is subdivided not into 12 but 41 equidistant intervals. From a conceptual point of view the Stahnke scale can be derived within the traditional theory since it is determined by only two constraints, namely to find good approximations for the major third and the harmonic seventh. The Bohlen–Pierce scale is beyond the classical realm as it is characterized by three intervals (3/1, 5/1, 7/1). In the next section we present a scale characterizable by even 5 constraints. Actually, we found this example just by chance. The remaining part of this article is mainly devoted to the problem, how multi–dimensional constraints may be explored systematically.

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[2]The Stahnke scale can be derived from the transcendental equations $5/4 = s^{(4-\mu)/12}$ and $7/4 = s^{(10+\mu)/12}$. They are solved by the numbers

$$s = \left(\frac{7}{4}\right)^{6/7} \left(\frac{5}{4}\right)^{6/7} = 1.95606847\ldots, \quad \mu = 7\frac{\ln(7/4) - \ln(5/4)}{\ln(7/4) + \ln(5/4)} - 3 = 0.00898\ldots$$

The solution μ is a transcendental number contrary to the number s which is simpler (algebraic number). The number μ characterizes the deviation of the intervals 4 and 10 of the Stahnke scale from the frequency ratios 5/4 and 7/4, respectively. The deviation corresponds to $1200 \log_2 s^{\mu/12}$ cent $= 100\mu \log_2 s$ cent $\approx$ 0.869 cent. The smallness of μ is a kind of a magic.

[3]The 6th harmonic corresponds nicely to the interval n =32, see Table 1.

[4]The relevant equation $3 = 2^x$ can be solved by a simple ratio x if the continued fraction expansion $\log_2 3 = \left\{\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{8}{5}, \frac{19}{12}, \frac{65}{41}, \ldots\right\}$ is considered.

2. Euler scale

The Euler number $e = 2.71828\ldots$ obeys the equation $0 = 1 + e^{i\pi}$ which is built of the most important mathematical numbers ($i = \sqrt{-1}$ is the *imaginary unit* solving $0 = 1 + i^2$).

The exponential function e^x can be represented by an infinite series

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \ldots$$

The series is rapidly converging, i.e. only a finite number of terms are needed in practice. For example, if evaluated at $x = 1$ only the first 9 terms are sufficient to determine e to 5 decimal places. The exponential function appears in many laws of nature.

To begin with we note

$$e^{7/10} \approx 2.014, \quad e^{11/10} \approx 3.004, \quad e^{16/10} \approx 4.953, \quad e^{24/10} \approx 11.02.$$

In the appendix, it is shown that the exponents 7/10, 11/10, 16/10, and 24/10 are best approximations of the 2nd, 3rd, 5th, and 11th harmonic

$$2 \simeq e^{n_2/N}, \quad 3 \simeq e^{n_3/N}, \quad 5 \simeq e^{n_5/N}, \quad 11 \simeq e^{n_{11}/N}$$

by integer intervals n_2, n_3, n_5, n_{11} and a *common* integer N.[5]

The Euler scale defined by

$$f = f_0\, e^{n/10}$$

appears to be a synthesis of the 12–tone ET scale and the Bohlen–Pierce scale. The 12–tone ET scale yields extremely good approximations of the 2nd harmonic and the 3rd harmonic. The Bohlen–Pierce scale represents well the harmonics 3, 5, and 7 (Table 1).

Upper partial tones corresponding well to *integer* intervals are marked in Table 1 by a star (*). The column of the Euler scale shows the largest number of stars and is, in a sense, the most 'harmonic' scale. This is mainly due to the fact that the Euler scale is approximating well not only the octave (2nd harmonic) but also the tritave (3rd harmonic) and, simultaneously, the prime–number valued harmonics 5 and 11.

The 12–tone ET scale approximates well the prime number-valued harmonics 2 and 3. The Bohlen–Pierce scale is approximating three prime number–valued harmonics (3, 5, 7).

If two harmonics are well approximated by two intervals, the product of the harmonics may also correspond to an integer interval. The error of the product is the sum of the errors of both intervals. See, for example, the interval $n = 31$ of the 12–tone ET scale that is the product of the 2nd harmonic and 3rd harmonic.

However, it may happen that the resulting error is too large. Although the 4th harmonic is the product of the 2nd harmonic with itself ($4 = 2 \times 2$), the interval $n = 14$ of the Euler scale is not well approximated (no star) since the error of the 2nd harmonic (≈ -0.07) leads to an absolute error of 0.14 which is beyond the cutoff value 0.1.[6]

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[5] By doubling the number of subintervals ($N = 20$) also the 7th harmonic can be well represented as $e^{39/20} \approx 7.029$.

[6] Measures for estimating the approximation error of a tonal scale are also considered in Section 4 and in the appendix.

$\frac{f}{f_0}$	12–tone ET $12\log_2\frac{f}{f_0}$	Bohlen-Pierce $13\log_3\frac{f}{f_0}$	Stahnke $12\log_{1.9560685}\frac{f}{f_0}$	Euler $10\ln\frac{f}{f_0}$
1	0 *	0 *	0 *	0 *
2	12 *	8.20	12.40	6.93 *
3	19.02 *	13 *	19.65	10.99 *
4	24 *	16.40	24.79	13.86
5	27.86	19.04 *	28.79	16.09 *
6	31.02 *	21.20	32.05 *	17.91 *
7	33.69	23.03 *	34.80	19.46
8	36 *	24.61	37.19	20.79
9	38.04 *	26 *	39.30	21.97 *
10	39.86	27.25	41.18	23.02 *
11	41.51	28.37	42.89	23.98 *

Table 1. Relation between the overtone spectrum and intervals n for different scales.
Notes: A star (*) indicates an almost integer interval number n (with an absolute error below the cutoff value < 0.1). For example, in the 12–tone ET (ET) scale the interval $n = 19$ corresponds almost identically to the 3rd harmonic. ($\ln = \log_e$ is the *natural* logarithm.)

n	$e^{n/10}$	$1200\log_2 e^{n/10}$ (cent)	$\frac{f}{f_0}$	$1200\log_2\frac{f}{f_0}$ (cent)	Difference (cent)
0	1	0	1/1	0	0
1	1.105	173	10/9	182	- 9
2	1.221	346	6/5	316	30
3	1.359	519	4/3	498	21
4	1.492	692	3/2	702	-10
5	1.649	866	5/3	884	-18
6	1.822	1038	9/5	1017	21
7	2.014	1212	2/1	1200	12
8	2.226	1385	2(10/9)	1382	3
9	2.460	1558	2(5/4)	1586	-28
10	2.718	1731	2(4/3)	1698	33
11	3.004	1904	2(3/2)	1902	2

Table 2. Relation between intervals n of the Euler scale and frequency ratios f/f_0.
Note: The last column shows the difference between the third and the fifth column.

A good approximation of harmonics out of ‘poor’ intervals may be realized because of a magic cancellation of errors. An example is given by the interval $n = 32$ of the Stahnke scale: the ‘poor’ intervals $n = 12, 20$ are rather neutral representations of the 2nd and 3rd harmonics. As already mentioned in the introduction the Stahnke scale was constructed to obtain an optimal approximation of the ‘fractional’ harmonics 5/4 (fifth) and 7/4 (seventh). Thus, good approximations to multiples of the fundamental frequency are possible just by chance.

The modified Stahnke scale $1.9560685^{n/30}$ splits the basic interval in 30 ET sections. It approximates the 2nd, 3rd and 7th harmonic very well, see Section 4 and the appendix.

More details on the interval structure of the Euler scale can be extracted from Table 2.

The intervals $n = 1, 2, \ldots 7$ correspond (up to differences of a few cents) to the frequency ratios of the minor tone (10/9), minor third (6/5), fourth (4/3), fifth (3/2), major sixth (5/3), minor seventh (9/5), and octave (2/1), respectively.

The interval $n = 11$ is just a tritave (2 cents up). In other words, the Euler scale divides the duodecime into 11 equal steps, nearly.[7] The intervals n=5 and n=6 are characterized by uneven frequency ratios which appear also in the Bohlen–Pierce scale as can be read off from $\log_3(5/3) = 6.045\ldots$ and $\log_3(9/5) = 6.955\ldots$.

The 2nd interval (346 cents) is almost in the middle of the major third (386 cents) and the minor third (316 cents). Triads consisting of the intervals $n = 0, 2, 4$ sound 'neutral', a little closer to minor than major triads.

The Euler scale is quite similar to equiheptatonic scales in traditional African and oceanic music where thirds are 'neutral' (between major and minor third and, in practice, fluctuating approx. ± 20 cent around the mean), fourths and fifths are quite exact, sevenths are also 'neutral' and octaves are often somewhat stretched on purpose [8], [9].

The 'Are'are from the Solomon Islands create complex equiheptatonic music on panpipes. An instrument maker starts a tuning process by measuring the length of a pipe with his fingers. Octaves are generated by doubling the size of a reference pipe or by cutting it in half. For the fine–tuning he uses his hearing while playing melodies and transposing them to different pitches [8]. As a measuring device the human ear has a much higher resolution power than a finger.[8]

The Euler number can be extracted from traditional African and oceanic music to an accuracy of a few per cent.

To show this assertion, we firstly note that from the tritave of the Euler scale the estimate

$$e \simeq (2^{1904/1200})^{10/11}$$

can be derived (see the third entry in the last row n=11 of Table 2).

There is no perfect tuning. Instruments in African music are considered as tuned in unison even if there are tuning fluctuations of up to a quarter tone [9]. Let us, therefore, assume that a traditional African instrument is tuned according to the Euler scale with a tuning error of ± 50 cents per interval. Then, we obtain for the accuracy of determining the Euler number using this instrument

$$e_{\text{non-European}} \simeq (2^{(1904\pm50)/1200})^{10/11} \simeq 2.647\ldots 2.790 \simeq e(1 \pm 0.026)$$

showing that the relative error is ∼2.6 %. Clearly, the smaller the tuning error the smaller the relative error. If the tuning error is only 10 cents, then, the relative error in extracting e reduces to 0.5 %. Moreover, the relative error is smaller, the higher the chosen interval. In light of this property, the assumption of a fixed tuning error per interval may not be fully realized in practice over the whole frequency range of interest.

The number $\pi = 3.14159\ldots$ characterizing the ratio of the circumference to the diameter of a circle, is transcendental like the Euler number. From the bible, the integer approximation $\pi_{\text{bible}} \simeq$ 30 cubit / 10 cubit = 3 can be extracted. A value of 3 was also used by the Babylonians and in ancient China. Since $\pi_{\text{bible}} \simeq \pi(1 - 0.045)$ the relative error of the ancient value is $\sim 4.5\%$

[7] The 10–tone ET Euler scale and the 11–tone ET Bohlen–Pierce scale differ only weakly. The equation $3 = e^x$ can be solved by a 'simple' ratio x if the continued fraction expansion $\ln 3 \simeq \left\{1, \frac{11}{10}, \frac{78}{71}, \ldots\right\}$ is considered. Thus, $3^{n/11} \simeq e^{(11/10)(n/11)} = e^{n/10}$. The 11–tone ET Bohlen–Pierce scale was already suggested in [2], footnote 26.

[8] Traditional Thai instruments are also tuned equiheptatonically, see [10] Chapter 15.

which is, quite remarkably, of the same order magnitude as the relative error just obtained for $e_{\mathrm{non-European}}$.

The number π can be determined by looking at circles. Such as the octave is a fundamental building block for periodic scales, the Euler number may musically be explored within the context of pseudo–octaves. Both numbers π and e, characterize periodic objects even though in different sensual spaces, π can be seen, e be heard.

3. The μEuclidean algorithm

Continued fractions[9] are used to find 'best approximations' for the ratio of two real numbers r_1, r_2 by a ratio of two integers m_1, m_2

$$\frac{r_1}{r_2} \simeq \frac{m_2}{m_1},$$

where the distance of the integers from the origin is limited. By rephrasing the *integer relation*

$$r_1 m_1 - r_2 m_2 \simeq 0$$

is obtained.

The ancient Euclidean algorithm allows to find 'small' non–vanishing integers m_1, m_2. It is an iterative algorithm based on the recursively defined numbers

$$r_{\nu+2} = r_\nu - \left\lfloor \frac{r_\nu}{r_{\nu+1}} \right\rfloor r_{\nu+1}, \quad \nu = 1, 2, 3 \ldots,$$

where $\lfloor r \rfloor$ is the floor function giving integer numbers by cancelling all digits after the decimal point, e.g. $\lfloor 2.98 \rfloor = 2$. By inserting all previous iterations integers $m_{1,\nu}, m_{2,\nu}$ can be determined such that a representation of the kind $r_{\nu+2} = r_1 m_{1,\nu} - r_2 m_{2,\nu}$ is obtained. The numbers r_ν decrease rapidly as ν increases since $r_{\nu+2} < r_\nu/2$ for all $\nu > 1$ which, in turn, follows from the inequalities $r - \lfloor r \rfloor < r/2$ for $r \geq 1$ and $\lfloor r \rfloor \leq r < \lfloor r \rfloor + 1$. If eventually $r_{\nu+2} = 0$ for some $\nu > 0$, the algorithm terminates and an integer relation is determined where $r_{\nu+1}$ is the *greatest common divisor* of r_1 and r_2.

The Euclidean algorithm and continued fractions are equivalent. Let the ratio r_1/r_2 be approximated by the n–th convergent, $r_1/r_2 \simeq [k_0, k_1, k_2, \ldots, k_n]$. The integers coefficients k_ν and the numbers r_ν of the Euclidean algorithm are related by $k_\nu = \lfloor r_{\nu+1}/r_{\nu+2} \rfloor$.

Finding integer relations for more than two real numbers

$$r_1 m_1 + r_2 m_2 + \cdots + r_k m_k \simeq 0$$

by a 'fast' algorithm is surprisingly difficult and has a long history initiated by Euler and Lagrange. Several partial solutions were suggested, e. g. for $n = 3$ [11]. The *Generalized Euclidean Algorithm* developed in 1977 was a breakthrough [12] and led eventually to the PSLQ algorithm [13].

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[9] Some properties of continued fractions are collected in the appendix.

The PSLQ algorithm is not identical to the Euclidean algorithm if restricted to the 'trivial' case of two real numbers ($k = 2$). This is due to the fact that the nearest integer function $\lfloor r \rceil = \lfloor r + 1/2 \rfloor$ is used instead of the floor function $\lfloor r \rfloor$.

This detail is of no relevance if identifying an integer relation for given real numbers is the main problem. However, we are interested in approximating real numbers by fractions and familiar 'best approximations' may not be recovered by the PSLQ algorithm.

Let us consider the traditional problem of approximating fifths characterized by the frequency ratio $f/f_0 = 3/2$. Within the 12–tone ET scale the ratio is well approximated by the 7th interval up to 0.3 % since $\log_2(3/2) = \frac{7}{12} \times 1.0028\ldots$. Within the 41–tone ET scale, the fifth is known to be approximated even better. The characteristic number $\log_2(3/2)$ can be approximated by the convergents $[0, 1, 1, 2, 2] = 7/12$ and $[0, 1, 1, 2, 2, 3] = 24/41$. The PSLQ algorithm finds the approximate integer relation $12 \cdot \log_2(3/2) - 7 \cdot 1 = 0.020 \cdots \simeq 0$ but not the improved approximation $41 \cdot \log_2(3/2) - 24 \cdot 1 = -0.017 \cdots \simeq 0$.

We present a modified PSLQ algorithm, called *μEuclidean algorithm*, that goes over into the Euclidean algorithm for $k = 2$, i.e. all approximations of the classical continued fraction expansion are recovered. The general strategy of the PSLQ algorithm is taken over, but an additional mirroring transformation is added in each iteration step. The proofs of the PSLQ algorithm rely in an essential way on the property of the nearest integer function to approximate real numbers 'optimally' by integer numbers since the distance $0 \le |r - \lfloor r \rceil| < 1/2$ is smaller, in general, than the distance $0 \le r - \lfloor r \rfloor < 1$. It turns out that the main assertions of the PSLQ algorithm survive. Also within the μEuclidean algorithm, the relevant intermediate numbers can 'optimally' be approximated by integer numbers, if the invariance of the PSLQ algorithm under mirror transformations is combined with the property that for any real number r the inequality $0 \le r - \lfloor r \rfloor \le 1/2$ or the inequality $0 \le -r - \lfloor -r \rfloor \le 1/2$ holds.

The *Generalized Euclidean Algorithm* replaces the single integer relation[10] $\langle r|m\rangle \simeq 0$ by a total of $k-1$ non–integer equations written as $\langle r|H = 0$ where H is a lower trapezoidal matrix whose column vectors span a hyperplane orthogonal to the vector $|r\rangle$. The matrix H is iteratively replaced by a sequence of matrices $H^{(\nu)} = (B^{(\nu)})^{-1} H Q^{(\nu)}$ where $B^{(\nu)}, Q^{(\nu)}$ ($\nu = 1, 2, \ldots$) are matrices that are constructed so that the norm $\|H^{(\nu)}\|$ is reduced as ν increases.[11] The matrices $B^{(\nu)}$ are integer valued and obey $\det B^{(\nu)} = \pm 1$, i.e. the inverse matrices ${B^{(\nu)}}^{-1}$ are also integer valued. The matrices $Q^{(\nu)}$ are orthogonal, i.e. they do not modify the scalar product between vectors. $Q^{(\nu)}$ are used in the algorithm to rotate ($\det Q^{(\nu)} = +1$) the row vectors of $H^{(\nu)}$ so that $H^{(\nu)}$ remains lower trapezoidal. If one of the components of $\langle r|B^{(\nu)}$ is zero for a finite ν, an integer relation $|m\rangle$ is stored in the corresponding column of $B^{(\nu)}$. If the algorithm does not terminate up to a certain ν, a 'small' integer relation does not exist and a lower bound for the smallest length of $|m\rangle$ can be determined from the norm of the matrix $H^{(\nu)}$.

There are many matrix norms.[12] Of relevance in the following is the Euclidean row norm $\|\cdot\|_{\infty,2}$

[10]Dirac's bra–(c)ket notation is used. The scalar product between a bra vector $\langle u| = (u_1, u_2, \ldots, u_k)$ and a ket vector $|v\rangle = (v_1, v_2, \ldots, v_k)^T$ reads $\langle u|v\rangle = u_1 v_2 + \cdots + u_k v_k$. In our setting, a bra vector is the transpose of a ket vector, $\langle u| = |u\rangle^T$.

[11]In the PSLQ algorithm the relevant norm ($\max_i |H^{(\nu)}_{ii}|$) is reduced monotonically, but not strongly monotonic.

[12]Let $p \ge 1$ be a real number. The *p–norm* of a k–dimensional vector $|r\rangle = (r_1, r_2, \ldots, r_k)^T$ is defined as $\||r\rangle\|_p = (|r_1|^p + |r_2|^p + \cdots + |r_k|^p)^{1/p}$. The case $p = 2$ corresponds to the Euclidean distance measure, $\||r\rangle\|_2 = \sqrt{\langle r|r\rangle}$. In the limit $p \to \infty$ the maximum norm is obtained, $\||r\rangle\|_\infty = \max_{1\le i\le k} |r_i|$. Let $q > p$, then $\||r\rangle\|_q \le \||r\rangle\|_p$ i.e. the topology induced by the p–norm is finer than the topology induced by the q–norm.
The *pq–norm* of a $(k \times l)$–matrix A can be defined as $\|A\|_{p,q} = \max_{|x\rangle\neq 0} \|A|x\rangle\|_p / \||x\rangle\|_q$ where $|x\rangle$ is an

which for a matrix A reads $\|A\|_{\infty,2} = \max_i \sqrt{\sum_j {A_{i,j}}^2}$.

In the PSLQ algorithm the lower trapezoidal matrix H is defined by the $k \times (k-1)$ components

$$H_{i,j} = \begin{cases} 0 & \text{if } i < j \\ s_{i+1}/s_i & \text{if } i = j \\ -r_i r_j/(s_{j+1}s_j) & \text{if } i > j \end{cases},$$

where $s_j^2 = \sum_{i=j}^{k} r_i^2$ are partial sums of squares of the input numbers. The absolute value of the components of H are restricted to the unit interval, more specifically $0 < H_{k-1,k-1} < \cdots < H_{2,2} < H_{1,1} < 1$ and $-1 < H_{i,j} < 0$ for all $i > j$. Let $\langle H_i|$ be the i-th row vector of H. Then $\langle H_i|H_i\rangle = 1 - (r_i/s_1)^2$. If the input numbers are ordered downwards, $r_1 > r_2 > \cdots > r_k > 0$, the lengths of the rows of H are ordered upwards and the row norm of H is given by the length of the last row, $\|H\|_{\infty,2} = \langle H_k|H_k\rangle^{1/2}$, and $H_{k-1,k-1} < |H_{k,k-1}|$. The column vectors of H are orthonormal, $H^T H = \mathbf{1}_{k-1}$.

Let $|m\rangle$ be a vector representing an integer relation, i.e. $\langle r|m\rangle = 0$. Then for any invertible matrix B and any orthogonal matrix Q, the following chain of relations holds

$$\begin{aligned} 1 &\le \|B^{-1}|m\rangle\|_\infty \\ &= \|B^{-1}HH^T|m\rangle\|_\infty \\ &\le \|B^{-1}H\|_{\infty,2}\ \|H^T|m\rangle\|_2 \\ &= \|B^{-1}H\|_{\infty,2}\ \||m\rangle\|_2 \\ &= \|B^{-1}HQ\|_{\infty,2}\ \||m\rangle\|_2, \end{aligned}$$

where we used that $\|H^T|m\rangle\|_2 = \langle m|HH^T|m\rangle^{1/2} = \langle m|m\rangle^{1/2} = \||m\rangle\|_2$ as $HH^T = \mathbf{1}_k - |r\rangle\langle r|/\langle r|r\rangle$. Consequently, there is a lower bound for the Euclidean length of the integer relation

$$\||m\rangle\|_2 \ge \frac{1}{\|H^{(\nu)}\|_{\infty,2}}.$$

This inequality is remarkable. It was used, for example, to find previously unknown relations between certain irrational numbers [14].

Just like the PSLQ algorithm we introduce a *reducing matrix* D with the difference that the nearest integer function $\lfloor\cdot\rceil$ is replaced by the floor function $\lfloor\cdot\rfloor$. D is determined recursively while updating H. Initially set $D = \mathbf{1}_k$ and $H' = H$ and fix a tiny real number $\varepsilon > 0$ to regularize numerical instabilities. For $i = 2, \ldots, k$ [step 1], for $j = (i-1)\ldots, 1$ [step -1] set $q = \lfloor (H'_{i,j} + \varepsilon)/H'_{j,j}\rfloor$, for $m = 1, \ldots, j$ [step 1] redefine $H'_{i,m} \to H'_{i,m} - qH'_{j,m}$, and for $m = 1, \ldots, k$ [step 1] redefine $D_{i,m} \to D_{i,m} - qD_{j,m}$.

The reducing matrix D is lower trapezoidal whose elements are integer with $D_{i,i} = 1$. The off–diagonal matrix elements of $H' = DH$ are smaller than the diagonal elements of H,

$$|H'_{i,j}| < |H'_{j,j}| = |H_{j,j}|, \quad \text{for all } i > j \tag{1}$$

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l–dimensional vector. Clearly, $\|A|x\rangle\|_p \le \|A\|_{p,q}\||x\rangle\|_q$.

Let $1/p + 1/q = 1$. Then $\|A\|_{\infty,p} = \max_{1\le i\le k} \|\langle a_i|^T\|_q$ where $\langle a_i|$ is the i–th row of the matrix A. The assertion follows from the Hólder inequality, $|\langle x|a\rangle| \le \||x\rangle\|_p\||a\rangle\|_q$ if $1/p + 1/q = 1$.

by virtue of the inequality $|x - \lfloor x/y \rfloor y| < |y|$.[13]

The reduction of the off–diagonal elements $H'_{i+1,i}$ can be improved. Let us introduce two matrices $S = \mathrm{diag}(s_1, s_2, \dots s_{k-1}, 1)$ and $S' = \mathrm{diag}(s_1, s_2, \dots s_{k-1})$ where $s_i = \pm 1$. Set initially $s_{k-1} = \mathrm{sign}(H_{k,k-1})$.[14] The remaining s_i are determined recursively. For $i = k-2, \dots, 1$ [step -1] solve $s_i s_{i+1} = \mathrm{sign}[\sin(2\pi H_{i+1,i}/H_{i,i})]$. (The recursion is well–defined since $H_{i,i} > 0$ for all $1 \le i \le k-1$ and for each iteration step.) Then, the secondary diagonal elements of the transformed matrix $DSHS'$ obey

$$|(DSHS')_{i+1,i}| \le \frac{1}{2}|H_{i,i}| \tag{2}$$

for all $i < k-1$.[15] The mirror transformation induced by the matrices S, S' is considerably improving the convergence behaviour of the μEuclidean algorithm introduced below.[16]

The proof is based on the fact that a mirrored number $-r$ fulfils $0 \le -r - \lfloor -r \rfloor \le 1/2$ whenever a real number r can only 'badly' be approximated by an integer number, $1/2 \le r - \lfloor r \rfloor \le 1$. The action of the reducing matrix D on the secondary diagonals of H is given by $(DH)_{i+1,i} = H_{i+1,i} - \lfloor H_{i+1,i}/H_{i,i} \rfloor H_{i,i}$. The matrices S, S' induce mirror transformations. The diagonals $H_{i,i}$ are left invariant and the secondary diagonals $H_{i+1,i}$ are multiplied with the proper signs, as $(SHS')_{i,j} = s_i s_j H_{i,j}$. □

Let P_r be a k–dimensional unit matrix whose rows r and $r+1$ are interchanged. P_r is a permutation matrix, $P_r^2 = \mathbf{1}_k$. It is used to transpose the rows r and $r+1$ of the matrix H.

The matrix

$$Q_r = \begin{pmatrix} \mathbf{1}_{r-1} & & & \\ & \beta/\delta & \lambda/\delta & \\ & \lambda/\delta & -\beta/\delta & \\ & & & \mathbf{1}_{k-2-r} \end{pmatrix}, \quad \delta = \sqrt{\beta^2 + \lambda^2}$$

is orthogonal. Let $\beta = H_{r+1,r}$ and $\lambda = H_{r+1,r+1}$. The transformation $H \to P_r H Q_r$ maps a lower trapezoidal matrix H into a lower trapezoidal matrix. The diagonal elements of $P_r H Q_r$ are positive since H is a lower trapezoidal matrix with positive diagonal elements and since Q_r is the product of a rotation and a mirror transformation ($\det Q_r = -1$).

The PSLQ algorithm breaks down for a certain class of integer relations due to a phenomenon not discussed before in the literature to the best of our knowledge.[17]

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[13]Without the rounding operation, i.e. for $q = H'_{i,j}/H'_{j,j}$, the transformed matrix H' would have been simply diagonal since then $DH = \mathrm{diag}(H_{11}, \dots, H_{k-1,k-1})$.

[14]$\mathrm{sign}(x) = 1$ for $x > \varepsilon$ and $\mathrm{sign}(x) = -1$ else, where $\varepsilon > 0$ is a tiny real number.

[15]The inequality does not hold for $i = k-1$ although the matrices S, S' can easily be modified to cover also this case. However, then the direct contact to the ancient Euclidean algorithm (k=2) is lost. But this detail is of no significant relevance as the Euclidean algorithm is converging fast, since $r_{\nu+2} < r_\nu/2$ as already mentioned.

[16]The inequality (2) can be extended to all off–diagonals by generalizing the mirror transformation implicitly: in the definition of the reducing matrix D the definition of q needs to be replaced by $q = (H'_{i,j} + \varepsilon)/H'_{j,j}$; $\mathrm{if}(q - \lfloor q \rfloor \le 1/2)\{q \to \lfloor q \rfloor\}; \mathrm{else}\{q \to -\lfloor -q \rfloor\};$. Then, the contact to the ancient Euclidean algorithm is lost, however, this deficiency can be repaired by using the q–replacement not for $i = k$, $j = k-1$. Although in this case, the μEuclidean algorithm is (almost) equivalent to the PSLQ algorithm with respect to the convergence behaviour, the matrix norm of H is also not decreasing strongly monotonic, see footnote 11.

[17]The phenomenon does not occur in the parallelized PSLQ (PPSLQ) algorithm [14] as its initial phase is modified essentially compared to the initial step of the PSLQ algorithm. We thank thank D. H. Bailey

The algorithm cannot be applied if the input number r_{k-1} is an integer multiple of r_k and, in the case $k > 2$, the parameter γ is sufficiently large.[18]

Proof Let $r_{k-1} = nr_k$ ($n = 1, 2, 3, \dots$). The essential off–diagonal element $H_{k,k-1}$ vanishes already after the initialization procedure of the PSLQ algorithm, since initially $H_{k-1,k-1} = r_k/s_{k-1}$ and $H_{k,k-1} = -r_{k-1}/s_{k-1}$ and, consequently, $q = \lfloor H_{k,k-1}/H_{k-1,k-1} \rceil = \lfloor -r_{k-1}/r_k \rceil = -n$, i.e. at the end of the initialization procedure where a reduction operation D is applied, $H_{k,k-1}$ is replaced by $(DH)_{k,k-1} = H_{k,k-1} - qH_{k-1,k-1} = -r_{k-1}/s_{k-1} + nr_k/s_{k-1} = 0$.

Let γ be sufficiently large. Then, $\gamma^{k-1}H_{k-1,k-1} \geq \gamma^i H_{i,i}$ for all $1 \leq i < k$. Consequently, the last two rows of H are transposed in the exchange step of the PSLQ algorithm with the result that now one of the diagonals of H is vanishing. The following reduction step in the main loop is not defined as the calculation of $q = \lfloor H_{k,k-1}/H_{k-1,k-1} \rceil$ suffers from a division by zero. ■

For example, the trivial integer relation $m_1 \cdot 1 + m_2 \cdot 1 = 0$ associated to the input numbers $r_1 = r_2 = 1$, cannot be solved by the PSLQ algorithm. A different counter example is given by the integer relation $m_1 \cdot 3 + m_2 \cdot 2 + m_3 \cdot 1 = 0$ and $\gamma > 5/\sqrt{14}$.

The PSLQ algorithm can be protected against this shortcoming by avoiding a reduction operation within the initial phase, as proposed by the following algorithm.

Moreover, the ordering of the *reduction step* and the *exchange step* in the main loop of the (P)PSLQ algorithm have to be interchanged to avoid a flaw in finding 'small' integer relations, as is explained below.

The *μEuclidean algorithm* consists of the following steps.

(0) Initialisation step. Calculate H from the numbers $r_1, r_2, \dots, r_k$, set $B = \mathbf{1}_k$, and fix a constant $\gamma > \sqrt{4/3}$ and a tiny constant $\varepsilon > 0$.
(1) Reducing step. Replace H by DH and B by BD^{-1}.
(2) Transposition step.
Select an integer number r such that $\gamma^r|H_{r,r}| \geq \gamma^i|H_{i,i}|$ for all $1 \leq i \leq k-1$. Replace H by P_rH and B by BP_r.
If $r < k-1$, make H lower trapezoidal by replacing H by HQ_r.
(3) Mirroring step. Replace H by SHS' and B by BS.
(4) Terminating step. The steps 1–3 are repeated unless one of the diagonals elements of H is vanishing, $H_{k-1,k-1} < \varepsilon$, or the maximum number of iterations have been carried out.

The lower bound on the parameter γ makes the transition step compatible with the matrix norm of H, just like for the PSLQ algorithm.

Proof Let the norm of H after the reducing step be given by the first row, $\|H\|^2_{\infty,2} = \langle H_1|H_1\rangle = \alpha^2$. The (square of the) length of the second row obeys $\langle H_2|H_2\rangle = H_{21}^2 + H_{22}^2 \leq \alpha^2/4 + \alpha^2/\gamma^2$ due to the previous mirroring step (i.e. the inequality is valid for all iterations of the main loop but the first, in general). A permutation P_1 is performed in the transition step if the length of the second row of H is smaller than the norm of H, $\alpha^2/4 + \alpha^2/\gamma^2 < \alpha^2$, i.e. if $\gamma > \sqrt{4/3}$. ■

The essential assertions about the PSLQ algorithm are also valid for the μEuclidean algorithm. The proofs can be taken over nearly literally and are omitted here. The basic prerequisites of the PSQL assertions are fulfilled also for the μEuclidean algorithm since, as already mentioned, due to the mirror transformation of Step 3 the relevant off–diagonal elements of the reduced matrix

for pointing the difference out to us.
[18]A breakdown occurs if $\gamma > (1 + r_{k-1}^2/r_k^2)^{1/2}$.

H are sufficiently small, $|(DSHS')_{i,i-1}| \leq (1/2)|H_{i,i}|$.

If the μEuclidean algorithm terminates after the νth iteration with $H^{(\nu)}_{k-1,k-1} < \varepsilon$, an integer relation $|m\rangle$ is stored in the last column of $B^{(\nu)}$ and

$$\||m\rangle\|_2 = \frac{1}{\left|H^{(\nu)}_{k,k-1}\right|}$$

is the Euclidean length of the integer relation.

The proof is identical to the proofs of Lemma 5 and Lemma 10 in [13] up to trivial modifications due to the property of the PSLQ algorithm to store the integer relation in the second–last column of $B^{(\nu)}$.

Let M be the length of the shortest integer relation of the numbers $r_1, \ldots, r_k$ ($M = \infty$ in the case of no integer relation). Then, there is an upper bound

$$\||m\rangle\|_2 \leq \gamma^{k-2} M$$

for the integer relation $|m\rangle$ stored in the last column of the matrix $B^{(\nu)}$. To discover 'small' integer relations the constant γ should be identified with the smallest of the allowed values, $\gamma = \sqrt{4/3} + \varepsilon$.

Proof If the μEuclidean algorithm terminates because an integer relation is found, the condition $\gamma^{k-1} H_{k-1,k-1} \geq \gamma^i H_{i,i}$ for all $1 \leq i \leq k-1$, is fulfilled. Then, by adopting Theorem 2 of [13] the assertion $\||B^{(\nu)}_k\rangle\|_2 \leq \gamma^{k-2} M$ can be extracted where in the last column $|B^{(\nu)}_k\rangle$ of $B^{(\nu)}$ an integer relation is stored. ■

The condition on the lowest diagonal element of H to be 'maximal' (i.e. $\gamma^{k-1} H_{k-1,k-1} \geq \gamma^i H_{i,i}$), is essential for the upper bound. Contrary to the μEuclidean algorithm, the (P)PSLQ algorithm may terminate without finding 'small' integer relations.

A simple counter example is given by the integer relation

$$3 \cdot m_1 + 1 \cdot m_2 + 6 \cdot m_3 = 0.$$

The (P)PSLQ algorithm terminates after the first loop and finds the integer relations $|m_a\rangle = (0, -6, 1)^T$ and $|m_b\rangle = (1, -3, 0)^T$. On the other hand, the μEuclidean finds the integer relation $|m_c\rangle = (-2, 0, 1)^T$ with the shortest length $M = \||m_c\rangle\|_2 = \sqrt{5}$. The solutions of the (P)PSLQ algorithm obey

$$\||m_a\rangle\|_2 = \sqrt{37} > \||m_b\rangle\|_2 = \sqrt{10} > \gamma M = \sqrt{6 + 2/3}$$

i.e. both of them violate the upper bound $\gamma^{k-2} M$.[19]

Besides the different ordering of the reducing step and transposition step in the main loop, the μEuclidean algorithm goes over into the PSLQ algorithm if (a) in the reduction step the

[19] There are infinitely many counter examples. Let, for example, $\alpha \geq 3$ be any integer. Applied to the integer relation $\alpha \cdot m_1 + 1 \cdot m_2 + 2\alpha \cdot m_3 = 0$, the (P)SLQ algorithm terminates after the first iteration and finds $|m_a\rangle = (0, -2\alpha, 1)^T$ and $|m_b\rangle = (1, -\alpha, 0)^T$. On the other hand, the μEuclidean algorithm identifies the smallest integer relation $|m_c\rangle = (-2, 0, 1)^T$. Since $|m_c\rangle$ is independent of α, both solutions $|m_a\rangle$, and $|m_b\rangle$ violate the upper bound the stronger the larger α.

floor function $\lfloor\cdot\rfloor$ is replaced by the nearest integer function $\lfloor\cdot\rceil$, (b) in the transposition step the matrix Q_r is replaced by

$$Q_r \begin{pmatrix} \mathbf{1}_r & & \\ & -1 & \\ & & \mathbf{1}_{k-2-r} \end{pmatrix},$$

and (c) the mirroring step is omitted.

We implemented the μEuclidean algorithm in the software package *Mathematica*. It turns out that the difference (a) may have a significant impact on the needed working precision if numerically non–trivial integer relations are analysed.

The algebraic number $\alpha = 3^{1/4} - 2^{1/4}$ is a solution of the equation $1 - 3860\alpha^4 - 666\alpha^8 - 20\alpha^{12} + \alpha^{16} = 0$. The μEuclidean algorithm applied to the input numbers $\langle r| = (1, \alpha, \alpha^2, \ldots, \alpha^{16})$ finds the integer coefficients of the polynomial equation after 1436 iterations with a working precision of 285 decimal digits (where $\varepsilon = 10^{-7}$). By skipping the mirroring step (3) of the algorithm and, instead, using the implicit mirror transformation presented in footnote 16, the integer relation is already found after 1426 iterations and, in doing so, the working precision can be reduced remarkably to 174 decimal digits (which is even 3 % better than the PSLQ algorithm).[20]

4. ET tunings and integer relations

Integer relations can be used to construct systematically equal tempered scales that approximate several frequency ratios simultaneously.

To begin with let us recall that within the usual equal–tempered scale there are relations between the major third, the fifth and the octave, for example the relations

$$2\,\text{major thirds} + 4\,\text{fifths} = 3\,\text{octaves}$$
$$1\,\text{major third} + 8\,\text{fifths} = 5\,\text{octaves}$$

can be deduced from the relations 12 fifths = 7 octaves ('circle of fifths') and 3 major thirds = 1 octave ('circle of thirds'). Taking into account that the third corresponds to the 4th interval, the fifth to the 7th interval and the octave to the 12th interval, we obtain the integer relations

$$(-3)\cdot 12 + 2\cdot 4 + 4\cdot 7 = 0$$
$$(-5)\cdot 12 + 1\cdot 4 + 8\cdot 7 = 0.$$

To make contact with the μEuclidean algorithm, the integer relations are represented as a matrix

[20] For comparison we also implemented the PSLQ algorithm in Mathematica and found that a working precision of 179 decimal digits is needed to find the solution (after 1426 iterations and with the termination condition that the absolute values of one of the entries of $\langle r|B^{(\nu)}$ or $H^{(\nu-1)}_{k-1,k}$ is smaller than $\varepsilon = 10^{-61}$). A working precision of 85 decimal digits as mentioned in [13] could be confirmed by an implementation of the PSLQ algorithm in the software package Maple [15]. The difference in the working precision should be due to the special error propagation handling within Mathematica.

equation

$$\begin{pmatrix} * & * & * \\ 12 & 4 & 7 \\ * & * & * \end{pmatrix} \begin{pmatrix} -5 & * & -3 \\ 1 & * & 2 \\ 8 & * & 4 \end{pmatrix} = \begin{pmatrix} * & * & * \\ 0 & * & 0 \\ * & * & * \end{pmatrix} .$$

The stars ($*$) mark currently undefined entries. It turns out that the matrix in the middle is just the matrix B of the μEuclidean algorithm defined in the previous section, and the matrix on the left–hand side is its inverse, i.e. the matrix equation is nothing but $B^{-1}B = \mathbf{1}$. See the result (5) where the $*$–entries of B^{-1} are replaced by numbers obtained from the μEuclidean algorithm.

We are looking for integer intervals $n_1, n_2, n_3, \ldots, n_k$ approximating well the harmonics or frequency ratios $p_1, p_2, p_3, \ldots, p_k$, i.e.

$$p_1 \simeq a^{n_1/N}, \quad p_2 \simeq a^{n_2/N}, \quad p_3 \simeq a^{n_3/N}, \quad \ldots \quad p_k \simeq a^{n_k/N}$$

where the positive real number a characterizes a non–octave interval and the integer N the number of its ET subintervals. We identify $p_1 = a$ or, equivalently, set $n_1 = N$ as determining N is part of the problem. By taking logarithms, we obtain the relations

$$r_1 = \log_a p_1 \simeq \frac{N}{N}, \quad r_2 = \log_a p_2 \simeq \frac{n_2}{N}, \quad \ldots \quad r_k = \log_a p_k \simeq \frac{n_k}{N}. \tag{3}$$

To solve this multi–dimensional approximation problem, we explore the integer relation ansatz $\langle r|m\rangle = 1 \cdot m_1 + r_2 m_2 + \ldots r_k m_k \simeq 0$.

A solution $N, n_2, \ldots, n_k$ of the approximation problem (3) is given by any of the rows of the matrix ${B^{(\nu)}}^{-1}$ obtained from the μEuclidean algorithm.

To motivate this assertion we consider the decomposition

$${B^{(\nu)}}^{-1} = |{B^{(\nu)}}^{-1}_1\rangle\langle r| + \Delta^{(\nu)} \tag{4}$$

where $|{B^{(\nu)}}^{-1}_1\rangle$ denotes the first column vector of ${B^{(\nu)}}^{-1}$. We analysed the decomposition for cases showing no integer relation between the numbers $r_1, \ldots, r_k$. In all cases it turned out that $\Delta^{(\nu)}$ tends to zero (rapidly) as the number of iterations ν is increased. However, the correction term $\Delta^{(\nu)}$ does not decrease strongly monotonic to zero, only in the mean. The length of $\langle r|B^{(\nu)}$ decreases as the number of iteration ν increases. Consequently, one of the eigenvalues of $B^{(\nu)}$ must tend to zero (but does not vanish since $\det B^{(\nu)} = \pm 1$) which, in turn, means that one eigenvalue of ${B^{(\nu)}}^{-1}$ increases. The eigenspaces of the eigenvalues of $B^{(\nu)}$ are non–degenerate if there is no 'hidden' symmetry between the input numbers. Given a decomposition (4) with a very small correction $\Delta^{(\nu)}$ only one eigenvalue of ${B^{(\nu)}}^{-1}$ is much larger than 1 and reads approximately $\langle r|{B^{(\nu)}}^{-1}_1\rangle$, the remaining (complex–valued) eigenvalues are close to zero (but none of them is vanishing because of the constraint $\det {B^{(\nu)}}^{-1} = \det B^{(\nu)} = \pm 1$). $\square$

In principle, the PSLQ algorithm may be used instead of the μEuclidean algorithm. However, in the case of two input numbers $(1, r)$ 'best approximations' obtained from continued fraction expansions may not be recovered as already mentioned in the previous section.

A tonal scale should not depend on the ordering of its defining numbers. An approximative solution $\langle {B^{(\nu)}}^{-1}_i|$ is called *stable* if it remains an approximative solution for any permutation of the input numbers $r_2, \ldots, r_k$ (not necessarily at the same iteration ν).

The position and the value of the first input number is fixed as $r_1 = 1$ used to determine the number N of ET subintervals. Thus, the case of two input numbers ($k = 2$) is stable per definition reflecting the fact that the continued fraction expansion of $r_1/r_2 = [k_0, k_1, k_2, \dots]$ can be used to write down directly the continued fraction expansion of the transposed ratio as $r_2/r_1 = [0, k_0, k_1, k_2, \dots]$ for $r_1 > r_2$.

The quality of an iteration ν can be estimated by determining how well the real numbers $N\langle r|$ are approximated by the integer numbers of the i–th row of ${B^{(\nu)}}^{-1}$.[21] We take the extreme point of view and consider the maximum norm of the difference [22]

$$\Delta_i^{(\nu)} = \left\| |r\rangle - \frac{1}{N} \left\langle {B^{(\nu)}}^{-1}_i \right|^T \right\|_\infty = \max_{2\le j\le k} \left| r_j - \frac{1}{N} {B^{(\nu)}}^{-1}_{i,j} \right| .$$

To explain the second equality we note that the number N is just the first entry of the i–th row of ${B^{(\nu)}}^{-1}$ and the first entry of $|r\rangle$ is the number 1 (by assumption), i.e. the difference of the first of the compared entries, $N r_1 - {B^{(\nu)}}^{-1}_{i,1}$, is always vanishing by construction.

Let us begin with the familiar approximation problem of the octave and the fifth, i.e. $\langle r| = (\log_{2/1}(2/1), \log_{2/1}(3/2)) = (1, \ln(3/2)/\ln 2) = (1, 0.585\dots)$. The third and fourth iteration yield[23]

$${B^{(3)}}^{-1} = \begin{pmatrix} 12 & 7 \\ -5 & -3 \end{pmatrix}, \quad |\Delta^{(3)}\rangle = \begin{pmatrix} 1.96 \\ 18.0 \end{pmatrix} \text{cent}$$

$${B^{(4)}}^{-1} = \begin{pmatrix} -41 & -24 \\ 12 & 7 \end{pmatrix}, \quad |\Delta^{(4)}\rangle = \begin{pmatrix} 0.48 \\ 1.96 \end{pmatrix} \text{cent}$$

respectively. The numbers in the rows of B^{-1} agree with the well–known result that the fifth and the octave can be best–approximated by $N = 5, 12$, or 41–tone ET scales where the fifth is given by the 3rd, 7th, and 24th interval, respectively. It should be noted that the negative numbers in some rows of B^{-1} are of no relevance since the numbers m_i of an the integer relation are only determined up to an overall sign. The larger the number N of ET subintervals the smaller are the components of $|\Delta^{(\nu)}\rangle$, i.e. the better is the quality of the tonal scale. However, if N becomes very large the tonal scale is of limited practical interest as adults can hardly distinguish successive pitch differences below 5 cents, except for several pitches played at the same time (chord) where interference effects allow for a much finer resolution.

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[21] All entries of a row are either positive of negative provided all input numbers have the same sign.

[22] In practical applications it is often convenient to represent errors in units of cent

$$\Delta_i^{(\nu)} = 1200 \log_2 a \max_{2\le j\le k} \left| r_j - \frac{1}{N} {B^{(\nu)}}^{-1}_{i,j} \right| \quad \text{cent.}$$

An approximation is called *reasonable* if $\Delta_i^{(\nu)} < 25$ cents, *good* if $\Delta_i^{(\nu)} < 15$ cents, and *impressive* if $\Delta_i^{(\nu)} < 5$ cents.

[23] The results of this section are obtained by an implementation of the μEuclidean algorithm in the programming language JavaScript (see http://www.hessling.net/mEuclid.html). JavaScript has a working precision of 16 decimal digits which is far more than sufficient for the calculations within this section (we set $\gamma = \sqrt{4/3} + \varepsilon$ and $\varepsilon = 10^{-10}$).

Taking also the major third into account, i.e. $\langle r| = (1, \log_2(5/4), \log_2(3/2))$, we obtain

$$B^{(9)^{-1}} = \begin{pmatrix} 34 & 11 & 20 \\ 12 & 4 & 7 \\ -53 & -17 & -31 \end{pmatrix}, \quad |\Delta^{(9)}\rangle = \begin{pmatrix} 3.93 \\ 13.69 \\ 1.41 \end{pmatrix} \text{cent} \tag{5}$$

showing that the 12–, 34–, and 53–ET scales yield approximations of increasing quality. However, the 12–tone ET scale is not stable, in harmony with the well–known problem that within this scale thirds are only poorly realized. The 34–tone scale and the 53–tone scale are impressive approximations.

In the 9–tone ET Chowning scale the octave corresponds the 13th interval since $9 \log_\chi 2 = 12.96\ldots$, see appendix for details about the non–octave scales analysed in this section. Inserting the numbers $\langle r| = (1, \log_\chi 2)$ into μEuclidean algorithm and the PSLQ algorithm, respectively, yields

$$B^{(3)}{}^{-1}_{\mu\text{Euclidean}} = \begin{pmatrix} -7 & -10 \\ 9 & 13 \end{pmatrix} \quad B^{(3)}{}^{-1}_{\text{PSLQ}} = \begin{pmatrix} -2 & -3 \\ 9 & 13 \end{pmatrix}$$

showing that the number of subintervals $N = 9$ is identified by both algorithms.

According to the continued fraction algorithm (see the appendix) also the 7–tone ET Chowning scale is approximating octaves well ($7 \log_\chi 2 = 10.08\ldots$), however the 7–tone ET approximation is not found by the PSLQ algorithm.

Representing the 3rd, 5th and 7th harmonics within the Bohlen–Pierce scale, i.e. $\langle r| = (\log_3 3, \log_3 5, \log_3 7) = (1, \ln(5)/\ln(3), \ln(7)/\ln(3))$, yields

$$B^{(14)^{-1}} = \begin{pmatrix} -13 & -19 & -23 \\ 101 & 148 & 179 \\ -170 & -249 & -301 \end{pmatrix}, \quad |\Delta^{(14)}\rangle = \begin{pmatrix} 6.53 \\ 1.97 \\ 1.25 \end{pmatrix} \text{cent.}$$

The quality of 13–tone ET scale (13–ET) is nearly impressive.

The 13–tone ET scale is also obtained and of the same quality if the characteristic ratios 3:5:7 of the Bohlen–Pierce scale, i.e. $\langle r| = (\log_3(3/1), \log_3(5/3), \log_3(7/5)) = (1, \ln(7/5)/\ln(3), \ln(5/3)/\ln(3))$, are plugged into the μEuclidean algorithm

$$B^{(9)^{-1}} = \begin{pmatrix} -13 & -4 & -6 \\ -271 & -83 & -126 \\ 101 & 31 & 47 \end{pmatrix}, \quad |\Delta^{(9)}\rangle = \begin{pmatrix} 6.53 \\ 0.06 \\ 1.25 \end{pmatrix} \text{cent.}$$

The quality of 101-ET is impressive.

Let $s = 1.9560685$ be the Stahnke number. The characteristic numbers $\langle r| = (1, \log_s(5/4), \log_s(7/4))$ of the Stahnke scale lead to

$$B^{(9)^{-1}} = \begin{pmatrix} 6 & 2 & 5 \\ 1332 & 443 & 1111 \\ -335 & -118 & -296 \end{pmatrix}, \quad |\Delta^{(9)}\rangle = \begin{pmatrix} 0.869 \\ 0.003 \\ 0.324 \end{pmatrix} \text{cent.}$$

The 6–tone ET scale suggested by this method seems to be in conflict with the choice $N = 12$ of the original Stahnke scale. However, after a multiplication with a factor of 2 the intervals of the Stahnke scale are recovered, see Table A1. The intervals obtained here for the harmonic seventh

(7/4) and the major third (5/4) agree with the results of the continued fraction expansion of the appendix. The impressive quality $|\Delta^{(9)}\rangle_1$=0.869 is due to the very construction of the Stahnke scale (see introduction).[24]

In the appendix, we show that the harmonics 2, 5, and 7 are approximated excellently by the 30–tone ET Stahnke scale. This result is also obtained here if the numbers $\langle r| = (1, \log_s 2, \log_s 5, \log_s 7)$ are plugged into the μEuclidean algorithm

$$B^{(19)^{-1}} = \begin{pmatrix} -30 & -31 & -72 & -87 \\ -331 & -342 & -794 & -960 \\ -28 & -29 & -67 & -81 \\ 2327 & 2404 & 5582 & 6749 \end{pmatrix}, \quad |\Delta^{(19)}\rangle = \begin{pmatrix} 1.40 \\ 0.15 \\ 8.63 \\ 0.02 \end{pmatrix} \text{cent.}$$

All four scales are stable. The 28–ET scale yields a good approximation.

The Euler scale approximates the 2nd, 3rd, 5th and 11th harmonics quite well as already shown in Section 2. This result is confirmed here since for the real numbers $\langle r| = (1, \ln 2, \ln 3, \ln 5, \ln 11)$ we obtain

$$B^{(19)^{-1}} = \begin{pmatrix} 10 & 7 & 11 & 16 & 24 \\ -94 & -65 & -103 & -151 & -225 \\ 274 & 190 & 301 & 441 & 657 \\ -61 & -42 & -67 & -98 & -146 \\ 1290 & 894 & 1417 & 2076 & 3093 \end{pmatrix}, \; |\Delta^{(19)}\rangle = \begin{pmatrix} 16.34 \\ 7.41 \\ 0.49 \\ 8.00 \\ 0.38 \end{pmatrix} \text{cent.}$$

The 10–ET scale is stable and separated by a large gap from the next stable choice 274-ET.

The 20–ET scale yields a reasonable approximation of the five lowest prime number harmonics 2, 3, 5, 7, and 11 by the intervals 14, 22, 32, 39, and 48 ($\Delta^{(35)} = 16.34$ scents).

The 10–ET scale approximates reasonably well ($\Delta^{(20)} = 20.0$ cents) *elementary ratios* 1/2, 3/2, 5/3, 11/5 consisting merely of the lowest prime numbers (except for 7).

There are several 20–ET scales representing well five ratios, e.g. 2/1, 3/1, 10/9, 11s/10, 11/7 ($\Delta = 12$ cent).

We have shown that the μEuclidean algorithm can be used conveniently to identify ET tonal scale that approximate several frequency ratios simultaneously. A measure is introduced to estimate the quality of the approximation.

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[24] The tonal content of the Stahnke scale, i.e. the set of numbers (6, 2, 5), can be obtained without referring to the Stahnke number by choosing, for example, the harmonic seventh (7/4) as the basis and determining approximations for the major third (5/4) and the octave (2/1) with respect to this basis. Inserting the numbers $\langle r| = (\log_{7/4}(7/4), \log_{7/4}(5/4), \log_{7/4}(2/1))$ we obtain

$$B^{(10)^{-1}} = \begin{pmatrix} -5 & -2 & -6 \\ 25 & 10 & 31 \\ -13 & -5 & 16 \end{pmatrix}, \quad |\Delta^{(10)}\rangle = \begin{pmatrix} 37.41 \\ 1.34 \\ 13.69 \end{pmatrix} \text{cent.}$$

The error associated to the first row of the matrix B^{-1} is quite large, roughly a quarter tone, since octaves a poorly approximated within the Stahnke scale. The numbers of the second line of the matrix (25,10,31) show the close connection between the Stahnke scale and the 31–tone ET system proposed by Huygens and others, see also footnote 28. The error is less than 2 cent. The solution (-13,-5,-16) in the last line of the matrix is not stable against a permutation of the input numbers.

5. Summary and outlook

The Euler scale can be understood as a harmonic synthesis of the well–tempered scale and the Bohlen–Pierce scale. It shares characteristic properties with traditional equiheptatonic scales in African and oceanic music.

The μEuclidean algorithm is an integer relation algorithm going over identically into the ancient Euclidean algorithm in the case of two dimensions. It can be used to approximate any finite set of real numbers by fractions with a common 'small' denominator and, in particular, to identify equidistant tunings which are pretty close to several given frequency ratios.

The special case of two dimensions is well explored. An essential question is which of the basic techniques and findings can be extended to higher dimensions.

A more refined analysis of multi–dimensional continued fractions needs a generalization of the concept of secondary convergents. The μEuclidean algorithm allows to approximate k real numbers by by $k+1$ integers. Multi–dimensional secondary convergents should be based on up to $2k$ integers in order to approximate each real number individually by a rational number. The integer numbers cannot be chosen freely but should be constrained by some currently unknown structure. A topologically motivated ansatz is to look for a convenient norm for minimising the distance between given real numbers and fractions with limited denominators.

In [16], the two–dimensional special linear transformations are used to explore well–formed scales which are shown to be related to secondary convergents. The transformations considered by Noll seem to be connected to the matrices $B^{(\nu)}$ of the μEuclidean algorithm.

The characterization problem of *diatonic scales* within well–formed scales is investigated in [17]. For every chromatic scale a generalized diatonic scale is determined. What are the conditions for specifying diatonic scales in higher dimensions and, even more, are some of the multi–dimensional diatonic scales singled out like the major and minor scales within the usual 12–tone scale?

The concept of well–formed scales can be generalized from two to three dimensions [18]. The essential idea is to implement algebraic structures in order to single out well–behaved scales. By combining algebraic and topological structures a characterization of multi–dimensional secondary convergents may be determined which, finally, might inspire some new music.

Appendix. Intervals and continued fractions

Let a be a positive real number characterizing a tonal scale. In the following

$a = 2$	octavian scale
$a = \frac{1}{2}(1+\sqrt{5}) = 1.618\ldots$	Chowning scale
$a = 3$	Bohlen–Pierce scale
$a = 1.9560685$	Stahnke scale
$a = e = 2.718\ldots$	Euler scale

are considered.

A frequency ratio $f/f_0 = p$ is called representable by an *integer* interval n if there exists an integer N such that

$$p = a^{(n+\varepsilon)/N}$$

where ε is a small real number ($|\varepsilon| < 1$).

For $p = a^m$ ($m = 1, 2, 3, \ldots$), a (trivial) solution exists if $n = mN$. In this case $\varepsilon = 0$.

Otherwise, consider the (equivalent) equation

$$\frac{n+\varepsilon}{N} = \frac{\log p}{\log a}$$

showing that the problem can be reduced to find a fractional approximation n/N of the real number $\log p/\log a$. Note that the ratio $\log p/\log a$ is independent of the basis of the logarithm. By identifying $\log = \log_a$ the ratio can be written $\log p/\log a = \log_a p$ since $\log_a a = 1$.

A generalization is to approximate *several* frequency ratios simultaneously by *one* tonal scale. The frequency ratios $p_1, p_2, \ldots, p_k$ are called representable by the *integer* intervals $n_{p_1}, n_{p_2}, \ldots, n_{p_k}$ if there exists a *common* integer N such that

$$p_1 = a^{(n_{p_1}+\varepsilon_{p_1})/N}, \quad p_2 = a^{(n_{p_2}+\varepsilon_{p_2})/N}, \quad \ldots, \quad p_k = a^{(n_{p_k}+\varepsilon_{p_k})/N}$$

where $\varepsilon_{p_1}, \varepsilon_{p_2}, \ldots, \varepsilon_{p_k}$ are small real numbers.

These equations are equivalent to

$$\frac{n_{p_1}+\varepsilon_{p_1}}{N} = \frac{\log p_1}{\log a}, \quad \frac{n_{p_2}+\varepsilon_{p_2}}{N} = \frac{\log p_2}{\log a}, \quad \ldots, \quad \frac{n_{p_k}+\varepsilon_{p_k}}{N} = \frac{\log p_k}{\log a}.$$

Again, the problem can be reduced to find fractional approximations of positive real numbers.[25]

Continued fractions. An important application of the theory of continued fractions is the approximation of real numbers by 'simple' rational numbers. Some results are summarized in the following. For more details and proofs, in particular, we refer to text books on number theory.

Any positive real number r can be represented by a continued fraction

$$\begin{aligned} r &= k_0 + \cfrac{1}{k_1 + \cfrac{1}{k_2 + \cfrac{1}{k_3 + \ldots}}} \\ &= [k_0, k_1, k_2, k_3, \ldots] \end{aligned}$$

where $k_0, k_1, k_2, k_3, \ldots$ are positive integers determinable by the Euclidean algorithm if applied to the numbers r, and 1. The n–th *convergent* of the number r is defined as a continued fraction terminated after the n–th entry

$$r \simeq [k_0, k_1, k_2, \ldots, k_n] = \frac{p_n}{q_n}$$

where p_n, q_n are integers. Even convergents are smaller than r and odd convergents larger

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \cdots \leq r \leq \cdots < \frac{p_3}{q_3} < \frac{p_1}{q_1}.$$

[25] The Editors pointed our attention to the authors of [21] who also explored 1–dimensional continued fractions for analysing non–standard tonal scales.

Reasonable approximations can also be obtained if the last entry of a convergent is reduced. The fractions

$$[k_0, k_1, \ldots, k_{n-1}, k] = \frac{k\, p_{n-1} + p_{n-2}}{k\, q_{n-1} + q_{n-2}}, \quad 1 \le k < k_n$$

are called the n–th secondary convergents of the number r.[26] The n–th convergent has $k_n - 1$ secondary convergents. Even secondary convergents are located between two successive even convergents

$$\frac{p_{2n}}{q_{2n}} < [k_0, \ldots, k_{2n+1}, 1] < \cdots < [k_0, \ldots, k_{2n+1}, k_{2n+2} - 1] < \frac{p_{2n+2}}{q_{2n+2}}$$

and odd secondary convergents are located between two successive odd convergents

$$\frac{p_{2n+3}}{q_{2n+3}} < [k_0, \ldots, k_{2n+2}, k_{2n+3} - 1] < \cdots < [k_0, \ldots, k_{2n+2}, 1] < \frac{p_{2n+1}}{q_{2n+1}}.$$

A *best approximation* of a real number $r > 0$ is defined as a fraction with a limited denominator. The following theorem is due to Lagrange:

If a fraction a/b is a best approximation of a real number $r > 0$, then a/b is a convergent or a secondary convergent. In other words, all best approximations of r are either convergents or secondary convergents.

If there exist integer approximations for two harmonics

$$p_1 = a^{(n_1+\varepsilon_1)/N}, \quad p_2 = a^{(n_2+\varepsilon_2)/N}$$

then for their product and their ratio there exist integer approximations in the same tonal scale since

$$p_1 p_2 = a^{(n_1+n_2+\varepsilon_1+\varepsilon_2)/N}, \quad p_1/p_2 \simeq a^{(n_1-n_2+\varepsilon_1-\varepsilon_2)/N}$$

or, equivalently,

$$\frac{n_1 + n_2 + \varepsilon_1 + \varepsilon_2}{N} = \frac{\ln(p_1 p_2)}{\ln a}, \quad \frac{n_1 - n_2 + \varepsilon_1 - \varepsilon_2}{N} = \frac{\ln(p_1/p_2)}{\ln a}$$

i.e. the harmonics $p_1 p_2$ and p_1/p_2 correspond to the intervals $n_1 \pm n_2$.

The error of the product (fraction) is given by the sum $\varepsilon_1 + \varepsilon_2$ (difference $\varepsilon_1 - \varepsilon_2$) of the individual errors ε_1 and ε_2.

A best approximation of a product or a fraction may not be sufficient practically, see footnote 6 in Section 2 and the example considered there.

Within the approach of best approximations it is sufficient to consider prime–number valued harmonics $p_1, p_2, \ldots, p_k$, since any harmonic n can be written as a product of prime numbers, $n = p_1^{n_1} \ldots p_k^{n_k}$ where the numbers $n_1, \ldots n_k$ stand for the multiplicities of the prime numbers.

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[26]The right hand side of the equation is well–defined if the quantities $p_{-2} = q_{-1} = 0,\ p_{-1} = q_{-2} = 1$ are introduced.

In practical applications the individual errors should not be too large. At least the condition

$$|\varepsilon_{p_1}| + |\varepsilon_{p_2}| + \cdots + |\varepsilon_{p_k}| < 1$$

should be fulfilled by the errors $\varepsilon_{p_1}, \varepsilon_{p_2}, \ldots, \varepsilon_{p_k}$ of the individual harmonics to avoid a non–unique interval mapping if products/ratios of the harmonics are considered.

Octavian scale. The 3rd harmonic can be approximated by fractional powers of the 2nd harmonic, $3 \approx 2^{n/N}$. To find good values for the fraction n/N consider the continued fraction expansion

$$\log_2 3 \simeq [1, 1, 1, 2, 2, 3, 1] \;\widehat{=}\; \left\{\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{8}{5}, \frac{19}{12}, \frac{65}{41}, \frac{84}{53}\right\}.$$

The sequence of fractions on the right hand side of the sign $\widehat{=}$ yield approximations of increasing accuracy to the number on the left hand side, $\log_2 3$. The octave may well be partitioned in $N = 5, 12, 41, 53$ intervals and the 3rd harmonic[27] corresponds to the interval 8, 19, 65 or 84, respectively. The choice $N = 12$ selects the well–tempered scale and $N = 5$ characterizes the 5–tone equal tempered scale often used in traditional African music.

Chowning scale. In his composition *Stria* (premiered in 1977) John Chowning used a scale based on the Golden Ratio

$$p = \chi^{n/9}, \quad \chi = \frac{1+\sqrt{5}}{2} = 1.618\ldots$$

see [19], [20].

From the continued fraction expansion

$$\log_\chi 2 \simeq [1, 2, 3, 1, 2] \;\widehat{=}\; \left\{\frac{1}{1}, \frac{3}{2}, \frac{10}{7}, \frac{13}{9}, \frac{36}{25}\right\}$$

it can be read off that a 9–tone ET scale yields a best approximation of octaves.

Bohlen–Pierce scale. The scale

$$p = 3^{n/13}$$

yields best approximations of the 3rd, 5th, and 7th harmonic since in the continued fraction expansions ($\log_3 3 = 1 = 13/13$)

$$\log_3 5 \simeq [1, 2, 6, 1, 1, 1] \;\widehat{=}\; \left\{\frac{1}{1}, \frac{3}{2}, \frac{19}{13}, \frac{22}{15}, \frac{41}{28}, \frac{63}{43}\right\},$$

$$\log_3 7 \simeq [1, 1, 3, 2, 1, 2] \;\widehat{=}\; \left\{\frac{1}{1}, \frac{2}{1}, \frac{7}{4}, \frac{16}{9}, \frac{23}{13}, \frac{62}{35}\right\}.$$

[27] The 3rd harmonic is nothing but a fifth (3/2) on top of an octave (2/1) since $3 = (2/1)(3/2)$.

there is the common denominator $N = 13$.

Stahnke scale. Let $s = 1.9560685$ be the Stahnke number. The first convergents of the 2nd, 5th, and 7th harmonics read

$$\log_s 2 \simeq [1, 30, 4] \mathrel{\widehat{=}} \left\{ \frac{1}{1}, \frac{31}{30}, \frac{125}{121} \right\},$$
$$\log_s 5 \simeq [2, 2, 1, 1, 32] \mathrel{\widehat{=}} \left\{ \frac{2}{1}, \frac{5}{2}, \frac{7}{3}, \frac{12}{5}, \frac{391}{163} \right\},$$
$$\log_s 7 \simeq [2, 1, 9, 34] \mathrel{\widehat{=}} \left\{ \frac{2}{1}, \frac{3}{1}, \frac{29}{10}, \frac{989}{341} \right\}.$$

There is a common denominator $N = 30$ leading to the best approximations

$$\log_s 2 \simeq \frac{31}{30} = \frac{n_2}{N}, \quad \log_s 5 \simeq \frac{12}{5} = \frac{72}{30} = \frac{n_5}{N}, \quad \log_s 7 \simeq \frac{29}{10} = \frac{87}{30} = \frac{n_7}{N}.$$

The quality of the approximations is very high because the denominators of the next convergents increase considerably.[28]

By taking ratios of the harmonics we find

$$\log_s \frac{5}{2^2} \simeq \frac{72}{30} - \frac{62}{30} = \frac{10}{30} = \frac{1}{3}, \quad \log_s \frac{7}{2^2} \simeq \frac{87}{30} - \frac{62}{30} = \frac{25}{30} = \frac{5}{6}$$

and both results are again best approximations because they appear in the continued fraction expansions of the major third and the harmonic seventh

$$\log_s \frac{5}{4} \simeq [0, 3, 148] \mathrel{\widehat{=}} \left\{ \frac{0}{1}, \frac{1}{3}, \frac{148}{445} \right\},$$
$$\log_s \frac{7}{4} \simeq [0, 1, 5, 36] \mathrel{\widehat{=}} \left\{ \frac{0}{1}, \frac{1}{1}, \frac{5}{6}, \frac{181}{217} \right\}.$$

In other words, the *modified* Stahnke scale $s^{n/30}$ approximates very well the 2nd harmonic (difference $\Delta = 0.3$ cent), the 5th harmonic ($\Delta = 1.4$ cent), and the 7th harmonic ($\Delta = 0.3$ cent).

Historically, the Stahnke scale (characterized by the exponent $n/12$ instead of $n/30$) was developed as a non–octaving scale which approximates optimally the major third and the harmonic seventh (see Section 1, the intervals $n = 4$ and $n = 10$ in Table A1, and footnote 24).

Euler scale. The first convergents of the natural logarithms of the first prime number–valued

[28] The basic interval $s^{1/30} = 1.02262\ldots\,(\widehat{=}\ 38.718$ cent) is almost identical to the basic interval $2^{1/31} = 1.02261\ldots\,(\widehat{=}\ 38.710$ cent) of the 31–tone ET scale considered already by Huygens (and others).

harmonics read

$$\begin{aligned}
\ln 2 &\simeq [0,1,2,3,1,6] \;\widehat{=}\; \left\{\frac{0}{1},\frac{1}{1},\frac{2}{3},\frac{7}{10},\frac{9}{13},\frac{61}{88}\right\}\\
\ln 3 &\simeq [1,10,7,9,2,2] \;\widehat{=}\; \left\{\frac{1}{1},\frac{11}{10},\frac{78}{71},\frac{713}{649},\frac{1504}{1369},\frac{3721}{3387}\right\}\\
\ln 5 &\simeq [1,1,1,1,1,3,1,1,1] \;\widehat{=}\; \left\{\frac{1}{1},\frac{2}{1},\frac{3}{2},\frac{5}{3},\frac{8}{5},\frac{29}{18},\frac{37}{23},\frac{66}{41},\frac{103}{64}\right\}\\
\ln 7 &\simeq [1,1,17,2,19,1] \;\widehat{=}\; \left\{\frac{1}{1},\frac{2}{1},\frac{35}{18},\frac{72}{37},\frac{1403}{721},\frac{1475}{758}\right\}\\
\ln 11 &\simeq [2,2,1,1,18,2] \;\widehat{=}\; \left\{\frac{2}{1},\frac{5}{2},\frac{7}{3},\frac{12}{5},\frac{223}{93},\frac{458}{191}\right\}
\end{aligned}$$

Thus, for the denominator $N = 10$ the best approximations of the 2nd, 3rd, 5th, and 11th harmonics read

$$\frac{n_2}{N} \simeq \frac{7}{10}, \quad \frac{n_3}{N} \simeq \frac{11}{10}, \quad \frac{n_5}{N} \simeq \frac{8}{5} = \frac{16}{10}, \quad \frac{n_{11}}{N} \simeq \frac{12}{5} = \frac{24}{10}$$

These findings are a mathematical explanation of the stars assigned to the harmonics 2, 3, 5, and 11 in Table 1 (Euler scale).

There are other 'common' denominators. For example, $N = 3387$ yields also a best approximations of the 2nd, 3rd, 5th, and 11th harmonics since they can be approximated by the convergents

$$\frac{n_2}{N} \simeq \frac{2}{3} = \frac{2258}{3387}, \quad \frac{n_3}{N} \simeq \frac{3721}{3387}, \quad \frac{n_5}{N} \simeq \frac{5}{3} = \frac{6545}{3387}, \quad \frac{n_{11}}{N} \simeq \frac{7}{3} = \frac{7903}{3387}.$$

Although the the choice $N = 3387$ makes sense mathematically it should be pointed out that it is almost useless practically since it is subdividing octaves into very tiny units ($1200 \log_2 e^{1/3387} \approx 0.511$ cent). Moreover, it leads to less accurate approximations of the 2nd, the 5th, and the 11th harmonic.

The 7th harmonic can be approximated by the 3rd secondary convergent

$$\ln 7 \simeq [1,1,9] = \frac{19}{10}.$$

However, the approximation is poor since $10 \ln 7 = 19.4591\ldots$ is located in the 'neutral' region between the intervals 19 and 20, see Table 1.

By purely considering convergents it is not possible to derive the result obtained with the μEuclidean algorithm that the prime–numbered harmonics 2, 3, 5, 11, and 7 can reasonably be represented within the 20–tone ET Euler scale. In this scale the 7th harmonic is represented by the 39th interval (see the fourth to last paragraph of Section 4 and footnote 5). The continued fraction [1,1,19]=39/20 does not belong to the convergents or secondary convergents of $\ln 7$.[29]

Convergents of the lowest intervals. Continued fraction expansions may be useful in identifying approximate frequency ratios for the lowest intervals. In Table A1 different tonal scales

[29] This observation motivated us to look for an alternative to 1–dimensional continued fractions and led us, eventually, to the μEuclidean algorithm.

n	12–tone ET $2^{n/12}$	Chowning $\chi^{n/9}$	Bohlen-Pierce $3^{n/13}$	Stahnke $s^{n/12}$	Euler $e^{n/10}$
1	$\left\{\frac{1}{1},\frac{17}{16},\frac{18}{17},\frac{89}{84}\right\}$	$\left\{\frac{1}{1},\frac{19}{18},\frac{77}{73},\frac{96}{91}\right\}$	$\left\{\frac{1}{1},\frac{12}{11},\frac{25}{23},\frac{37}{34}\right\}$	$\left\{\frac{1}{1},\frac{18}{17},\frac{37}{35}\right\}$	$\left\{\frac{1}{1},\frac{10}{9},\frac{11}{10},\frac{21}{19}\right\}$
2	$\left\{\frac{1}{1},\frac{9}{8},\frac{55}{49}\right\}$	$\left\{\frac{1}{1},\frac{9}{8},\frac{10}{9},\frac{69}{62}\right\}$	$\left\{\frac{1}{1},\frac{6}{5},\frac{13}{11},\frac{45}{38}\right\}$	$\left\{\frac{1}{1},\frac{9}{8},\frac{19}{17},\frac{85}{76}\right\}$	$\left\{\frac{1}{1},\frac{5}{4},\frac{6}{5},\frac{11}{9}\right\}$
3	$\left\{\frac{1}{1},\frac{6}{5},\frac{19}{16},\frac{25}{21}\right\}$	$\left\{\frac{1}{1},\frac{6}{5},\frac{7}{6},\frac{20}{17}\right\}$	$\left\{\frac{1}{1},\frac{4}{3},\frac{9}{7},\frac{58}{45}\right\}$	$\left\{\frac{1}{1},\frac{6}{5},\frac{13}{11},\frac{123}{104}\right\}$	$\left\{\frac{1}{1},\frac{3}{2},\frac{4}{3},\frac{27}{20}\right\}$
4	$\left\{\frac{1}{1},\frac{4}{3},\frac{5}{4},\frac{29}{23}\right\}$	$\left\{\frac{1}{1},\frac{5}{4},\frac{26}{21},\frac{135}{109}\right\}$	$\left\{\frac{1}{1},\frac{3}{2},\frac{7}{5},\frac{122}{87}\right\}$	$\left\{\frac{1}{1},\frac{4}{3},\frac{5}{4},\frac{494}{395}\right\}$	$\left\{\frac{1}{1},\frac{3}{2},\frac{91}{61}\right\}$
5	$\left\{\frac{1}{1},\frac{3}{2},\frac{4}{3},\frac{295}{221}\right\}$	$\left\{\frac{1}{1},\frac{4}{3},\frac{13}{10},\frac{17}{13}\right\}$	$\left\{\frac{1}{1},\frac{2}{1},\frac{3}{2},\frac{29}{19}\right\}$	$\left\{\frac{1}{1},\frac{4}{3},\frac{37}{28},\frac{41}{31}\right\}$	$\left\{\frac{1}{1},\frac{2}{1},\frac{3}{2},\frac{5}{3},\frac{28}{17}\right\}$
6	$\left\{\frac{1}{1},\frac{3}{2},\frac{7}{5},\frac{17}{12}\right\}$	$\left\{\frac{1}{1},\frac{3}{2},\frac{4}{3},\frac{7}{5}\right\}$	$\left\{\frac{1}{1},\frac{2}{1},\frac{3}{2},\frac{5}{3},\frac{88}{53}\right\}$	$\left\{\frac{1}{1},\frac{3}{2},\frac{4}{3},\frac{7}{5}\right\}$	$\left\{\frac{1}{1},\frac{2}{1},\frac{9}{5},\frac{11}{6},\frac{20}{11}\right\}$
7	$\left\{\frac{1}{1},\frac{3}{2},\frac{442}{295}\right\}$	$\left\{\frac{1}{1},\frac{3}{2},\frac{13}{9},\frac{16}{11}\right\}$	$\left\{\frac{1}{1},\frac{2}{1},\frac{9}{5},\frac{47}{26}\right\}$	$\left\{\frac{1}{1},\frac{3}{2},\frac{34}{23},\frac{71}{48}\right\}$	$\left\{\frac{2}{1},\frac{145}{72}\right\}$
8	$\left\{\frac{1}{1},\frac{2}{1},\frac{3}{2},\frac{8}{5},\frac{19}{12}\right\}$	$\left\{\frac{1}{1},\frac{2}{1},\frac{3}{2},\frac{20}{13}\right\}$	$\left\{\frac{1}{1},\frac{2}{1},\frac{57}{29}\right\}$	$\left\{\frac{1}{1},\frac{2}{1},\frac{3}{2},\frac{11}{7}\right\}$	$\left\{\frac{2}{1},\frac{9}{4},\frac{20}{9},\frac{69}{31}\right\}$
9	$\left\{\frac{1}{1},\frac{2}{1},\frac{5}{3},\frac{37}{22}\right\}$	$\left\{\frac{1}{1},\frac{2}{1},\frac{3}{2},\frac{5}{3},\frac{8}{5}\right\}$	$\left\{\frac{2}{1},\frac{15}{7},\frac{77}{36}\right\}$	$\left\{\frac{1}{1},\frac{2}{1},\frac{3}{2},\frac{5}{3}\right\}$	$\left\{\frac{2}{1},\frac{5}{2},\frac{27}{11},\frac{32}{13}\right\}$
10	$\left\{\frac{1}{1},\frac{2}{1},\frac{7}{4},\frac{9}{5},\frac{16}{9}\right\}$	$\left\{\frac{1}{1},\frac{2}{1},\frac{5}{3},\frac{12}{7}\right\}$	$\left\{\frac{2}{1},\frac{7}{3},\frac{149}{64}\right\}$	$\left\{\frac{1}{1},\frac{2}{1},\frac{5}{3},\frac{7}{4}\right\}$	$\left\{\frac{2}{1},\frac{3}{1},\frac{8}{3},\frac{11}{4}\right\}$
11	$\left\{\frac{1}{1},\frac{2}{1},\frac{15}{8},\frac{17}{9}\right\}$	$\left\{\frac{1}{1},\frac{2}{1},\frac{9}{5},\frac{551}{306}\right\}$	$\left\{\frac{2}{1},\frac{3}{1},\frac{5}{2},\frac{33}{13},\frac{38}{15}\right\}$	$\left\{\frac{1}{1},\frac{2}{1},\frac{11}{6},\frac{13}{7}\right\}$	$\left\{\frac{3}{1},\frac{721}{240}\right\}$

Table A1. The first convergents of the intervals $n = 1, 2, \ldots, 11$ for different scales. $\chi = (1 + \sqrt{5})/2 = 1.618\ldots$ is the Golden Ratio, $s = 1.9560685$ the Stahnke number, and $e = 2.718\ldots$ the Euler number. The first four convergents are noted, in general. In some cases less (or more) convergents are listed to indicate that a continued fraction expansion is converging well (or only slowly).

are compared. The accuracy of a convergent is the higher the larger the difference is between its denominator and the denominator of the next convergent. For example, the approximation 3/2 for the 7th interval of the 12–tone (ET) scale is very good since the next convergent 442/295 has a much larger denominator.

It should be noted that the fractions f/f_0 associated to the intervals n in Table 2 appear also in the list of convergents of the Euler scale in Table A1.

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